REVIEW OF FORWARD BACKWARD SWEEP METHOD FOR BOUNDED AND UNBOUNDED CONTROL PROBLEM WITH PAYOFF TERM

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ABSTRACT*— This paper is review of forward backward sweep method for bounded and unbounded control problem with payoff term, as someone want maximize function at specific time or mostly at final time. We used RK, Euler, trapezoidal techniques to solve such problems with different step sizes.*

Keywords— Forward-Backward Sweep Method (FBSM), Optimal Control Problems, BVPs, Payoff Term

1. INTRODUCTION

In mathematics, dynamical systems that have some constraints on it, then finding out the inputs that optimize (maximize or minimize) a particular cost functional according to requirement and satisfying the given constraints is called Optimal Control.

Origin of Optimal control is associated to the concept of calculus of variations. There are some important names that contribute to the theory of optimal control include William Hamilton, Isaac Newton, Johann Bernoulli, Adolph Mayer, Carl Jacobi, Andrien Legendre, Leonhard Euler, Ludovic Lagrange, , Karl Weierstrass, and Oskar Bolza. After Second World War, mathematicians use their mathematical theories in defence analyses and improve methods for the solutions of problems like minimize time interception problems for fighter jet, which later recognized as optimal control problem. In 1950 for aircraft Angelo Miele explored optimal control problems with control bounds. He also explains this bounded control concept for rockets in a series of papers from 1950 on. In the 20th century concept of optimal control improve when Lev Pontryagin and his co-worker present the minimum principal, Richard Bellman formulates the dynamic programming and Rudolf Kalman introduces the linear quadratic regulator [1].

Typically optimization problems have basic three elements. The first is the objective functional which is to be maximized or minimized. Examples are; predicted profit on any investment, the time of arrival of any motor vehicle at given target, sending a rocket to the moon with limited amount of fuel. The next part is group of those variables, whose values can be used to optimize the objective functional. For example, the amount of stock to be bought or sold, the path to be monitored by a vehicle through a traffic system. The last part of an optimization problem is a set of constraints, which restrict the values of variables. For example, employee of any factory cannot work more than t hours per day and only four worker can use the same instrument at a time, the pipeline's maximal fuel amount is x[2].

After the hard work of several generations of mathematicians they established strong background for further research and finds its applications in many scientific fields like Management sciences and Biomedical [3]. For the solution of optimal control problem the FBSM is indirect numerical method which is easy for computer programming and also execute quickly. For two-point boundary value problem, Enright and Muir use implicit and explicit Rang-e-Kutta methods with some flavour of FBSM. In order to numerically solve a problem with the addition of bounds and payoff term, we will do modification in FBSM algorithm [4].

2. MATERIALS AND METHODS

Payoff Term

Sometimes, in addition to maximize (or minimize) term over the entire time interval, we want to maximize a function value at specific point in time, specially, at the end of time interval. For example; in cancer model we want to minimize the tumour cells at the final stage .Minimize the number of infected people at the final time in an epidemic model. So term $\phi(x(t_1))$ called payoff term. It is sometimes referred as salvage term [5].

Optimal Control Problem with Payoff Term

Suppose we have the optimal control problem with payoff term, but without control bounds

$$\max\ [\phi(x(t_1)) + \int_{t_0}^{t_1} f(t,x(t),u(t)) \quad ---(1)$$

Subject to

$$x'(t) = g(t,x(t),u(t))\ ,\ x(t) = x_0$$

To attain a genuine solution in optimal control, many problems require bounds on control. In order to solve above optimal problems with control bounds, we have

$$a \le u(t) \le b$$

Where a and b are fixed, real constants and $a < b$

The Maximum Principal and Necessary Conditions

For the solution of optimal control problem the principal method resolves a set of necessary conditions that an optimal control and the consistent state equation must satisfy. At this stage it is essential to realize the logical and reasonable difference between necessary conditions and sufficient conditions [6].

Necessary Conditions: if $u^*(t)$, $x^*(t)$ are optimal, then the following conditions hold…

Sufficient Conditions: if $u^*(t)$, $x^*(t)$ satisfy the following condition, then $u^*(t)$, $x^*(t)$ are optimal.

Optimal control problem with control constraints

$$\max\ [\phi(x(t_1)) + \int_{t_0}^{t_1} f(t,x(t),u(t))$$

Subject to

$$x'(t) = g(t,x(t),u(t))\ ,\ x(t) = x_0$$

Then there exists a piecewise differentiable adjoint variable λ. The first order necessary condition is famous as the Maximum Principle. Firstly, for the solution of an Optimal Control problem, need to change the constrained optimization problem into a unconstrained problem, and the consequential function is known as the Hamiltonian function denoted as H. The following is an outline of how this theory can be applied to solve the simplest problems.

1) Form the Hamiltonian for the problem:

$$H(t,x,u,\lambda) = f(t,x,u) + \lambda g(t,x,u)$$

Where λ is known as adjoint variable, which is similar to the Lagrange multiplier. It attaches the differential equation information onto the maximization of the objective functional.

2) Write the adjoint differential equation, transversality boundary condition, and the optimality condition. Now there are three unknowns, u^*, x^* and λ.

$$\frac{\partial H}{\partial u} = 0\ at\ u^* \Rightarrow f_u + \lambda g_u = 0$$

In case of bounded control$a \le u(t) \le b$, we have discussed three cases.

$$\begin{cases} u^* = a & if\ \dfrac{\partial H}{\partial u} < 0 \\ a \le u^* \le b & if\ \dfrac{\partial H}{\partial u} = 0 \\ u^* = b & if\ \dfrac{\partial H}{\partial u} > 0 \end{cases}$$

$$\frac{\partial H}{\partial x} \Rightarrow\ \lambda' = -(f_x + \lambda g_x) \text{ (Adjoint equation)}$$

$$\lambda(t_1) = 0 \text{ (Transversality condition)}$$

When problem involve payoff term, then

$$\lambda(t_1) = \phi'(x(t_1))$$

We are given the dynamics of the state equation

$$x'(t) = g(t,x,u)\ ,\ x(t_0) = x_0$$

3) Try to remove u^* by using the optimality equation $H_u = 0$, i.e. Solve for in term of x^* and λ.

4) Solve the two differential equation for x^* and λ with two boundary conditions, substituting u^* in the differential equation with the expression for the optimal control from the previous step.

5) After finding the optimal state and adjoint, solve for the optimal control.

Algorithm of FBSM

A rough outline of the algorithm for the Forward-Backward Sweep method is given below. Here, $\vec{x} = x_1, \ldots, x_{N+1}$ and $\vec{\lambda} = \lambda_1, \ldots, \lambda_{N+1}$ are the vector approximations for the state and adjoint.

Step-1. Create an initial guess for $\vec{u}$ over the interval.

Step-2. Using the initial condition $x(t_0) = x_0$ and the values for $\vec{u}$, solve $\vec{x}$ forward in time regarding to its differential equation.

Step-3. Using the transversality condition $\lambda(t_1) = 0$ or $\lambda(t_1) = \phi'(x(t_1))$ and the values for $\vec{u}$ and $\vec{x}$, solve $\vec{\lambda}$ backward in time as given to its equation in the optimality system.

Step-4. Update $\vec{u}$ by entering the new $\vec{x}$ and $\vec{\lambda}$ values into the characterization of the optimal control.

The idea exploited by the FBSM can be seen in the way one of the equations is solved in a forward direction and the other is solved backwards with updates from the first. The method continues by using these new updates. This algorithm needs modification if we solve optimal control problem with payoff term. We will solve the above optimality system with FBSM by using the following numerical schemes:

1. The Euler's Scheme
2. The Trapezoidal Scheme
3. The Rung-e-Kutta Scheme

1. Euler Method

We approximate the solution of the two point BVP on a uniform line mesh by Euler's implicit method.

We set $h = \frac{t_1}{N}$ and choose the mesh points $t_i = (i-1)h, \quad i = 1,2, \ldots, N+1$

Applying the Implicit-Euler method to resulting optimality system, starting for x at $t_1 = 0$ and for λ at $t_{N+1} = t_1$.

We obtain the following system of non-linear equations.

$$x_{i+1} = x_i + h[g(t_{i+1}, x_{i+1}, u_{i+1})] --- (2)$$
$$x_1 = x_0, \; i = 1,2, \ldots, N$$
$$\lambda_i = \lambda_{i+1} - h(f_{x_i} + \lambda_i g_{x_i}) --- (3)$$
$$\lambda_{N+1} = 0 \text{ Or } \phi'(x(t_1)), \; i = 1,2, \ldots, N$$

This is a system of equations for the 2N n vector

$$x_i \approx x(t_i), = 1,2, \ldots, N+1,$$
$$\lambda_i \approx \lambda(t_i), i = 1,2, \ldots, N.$$

2. Trapezoidal Method

Since the Euler method is only a first order method, we also tried the trapezoidal scheme, which is a second order scheme [7]. However, the trapezoidal scheme is A-stable whereas the Euler scheme is A-stable and L-stable.

Applying the Trapezoidal scheme to the resulting optimality system, we obtain the following system of equations.

$$x_{i+1} = x_i + \frac{h}{2}[g(t_i, x_i, u_i) + g(t_{i+1}, x_{i+1}, u_{i+1})] --- (4)$$
$$x_1 = x_0, \; i = 1,2, \ldots, N$$
$$\lambda_i = \lambda_{i+1} - \frac{h}{2}[(f_{x_i} + \lambda_i g_{x_i}) + (f_{x_{i+1}} + \lambda_{i+1} g_{x_{i+1}})] --- (5)$$
$$\lambda_{N+1} = 0 \text{ Or } \phi'(x(t_1)), \; i = 1,2, \ldots, N$$

3. The Rung-e-Kutta Scheme

One member of the family of Rung-e-Kutta methods is often stated to as "RK4" or "classical Rung-e-Kutta method". Applying the RK4 scheme to the resulting optimality system, we obtain the following system of equations.

$$x_{i+1} = x_i + \frac{h}{6}[g(t_i, x_i, u_i) + 2g(t_{i+1}, x_{i+1}, u_{i+1}) + 2g(t_{i+2}, x_{i+2}, u_{i+2}) + g(t_{i+}$$
$$x_1 = x_0, \; i = 1,2, \ldots, N$$
$$\lambda_i = \lambda_{i+1} - \frac{h}{6}[(f_{x_i} + \lambda_i g_{x_i}) + 2(f_{x_{i+1}} + \lambda_{i+1} g_{x_{i+1}}) + 2(f_{x_{i+2}} + \lambda_{i+2} g$$
$$\lambda_{N+1} = 0 \text{ Or } \phi'(x(t_1)), \; i = 1,2, \ldots, N$$

Now, we take an optimal control problem as test problems, which we solve by FBSM using Euler and Trapezoidal schemes, and executed by MATLAB.

3. RESULTS

Optimal control problem without control constraints

Consider the problem

$$\max \int_0^2 \left(x(t) - \frac{1}{2} u^2(t) \right) dt + 2x(2)$$

Subject to

$$x'(t) = -x(t) + u(t), x(0) = 2$$

Construct Hamiltonian for this problem

$$H = x(t) - \frac{1}{2} u^2(t) + \lambda(-x + u)$$

The payoff term $\phi = 2x(2)$ is not included in the Hamiltonian, so only change in necessary conditions is in the transversality condition. Specifically, since $\phi(x) = 2x(2)$ and $\phi' = 2$. Then, the adjoint calculation yields

$$\lambda'(t) = -\frac{\partial H}{\partial x} = -1 + \lambda, \quad \lambda(2) = \varphi' = 2$$

The optimality conditions gives

$$\frac{\partial H}{\partial u} = 0 \; at \; u^* \Rightarrow -u + \lambda = 0$$
$$u^* = \lambda$$

We take TOL: 0.001

TABLE1: OPTIMAL CONTROL VALUES EVALUATED BY RK-METHOD, EULER AND TRAPEZOIDAL METHOD AT H=0.002

Selected values of Time	FBSM with Euler's	FBSM with Trapezoidal	FBSM with R-K
0.0000	1.1350	1.1342	1.1342
0.0080	1.1361	1.1353	1.1353
0.0380	1.1403	1.1395	1.1395
0.0980	1.1490	1.1481	1.1481
0.1260	1.1532	1.1524	1.1524
0.3080	1.1839	1.1830	1.1830
0.8160	1.3058	1.3048	1.3048
1.0200	1.3750	1.3740	1.3740
1.2040	1.4507	1.4497	1.4497
1.8220	1.8356	1.8351	1.8351
1.9500	1.9495	1.9493	1.9493
2.0000	1.9980	1.9880	1.9880

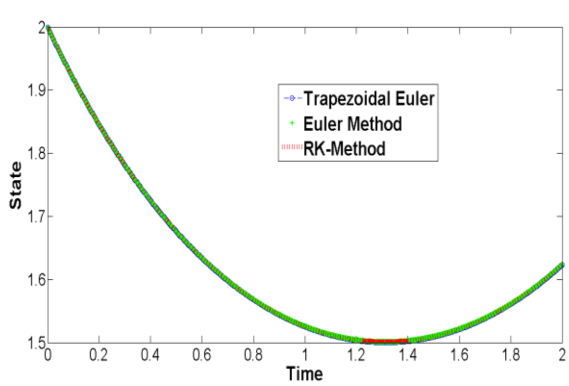

Fig. 1: Optimal state values at h=0.002

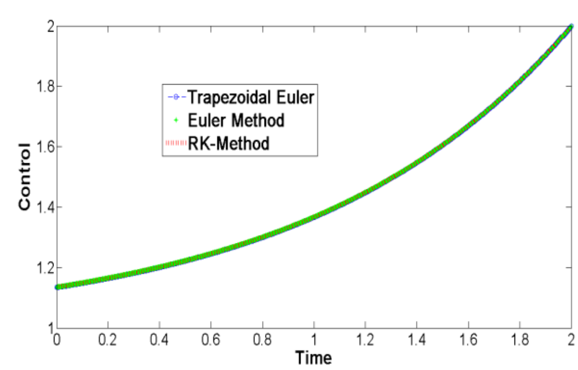

Fig. 2: Optimal Control values at h=0.002

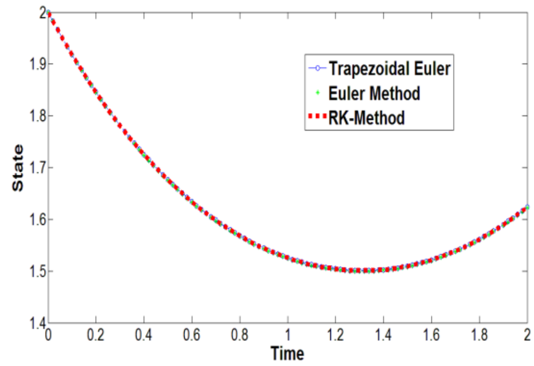

Fig. 3: Optimal state values at h=0.02

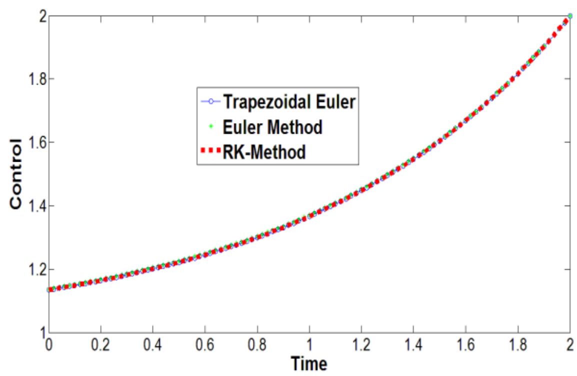

Fig. 4: Optimal Control values at h=0.002

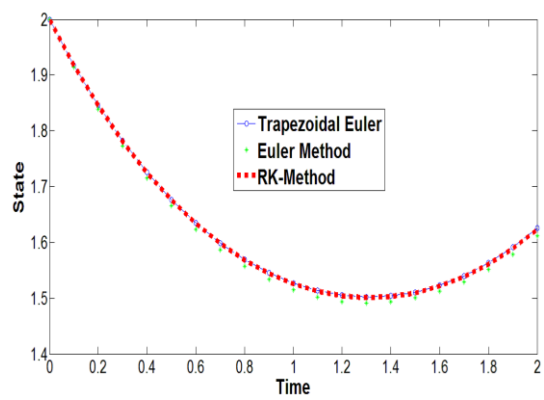

Fig. 5: Optimal state values at h=0.1

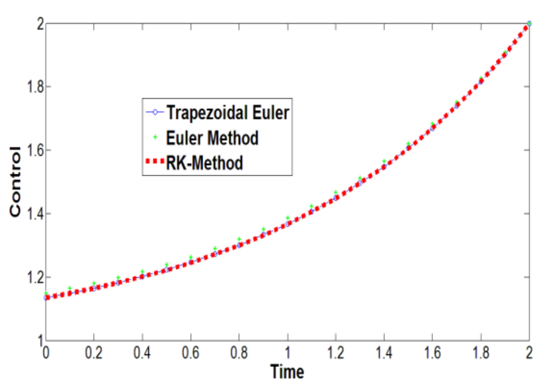

Fig. 6: Optimal Control values at h=0.1

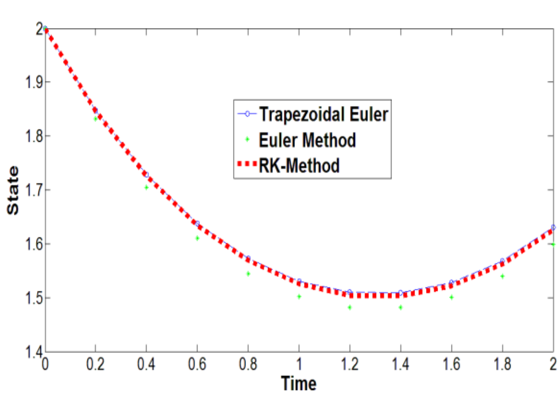

Fig. 7: Optimal state values at h=0.2

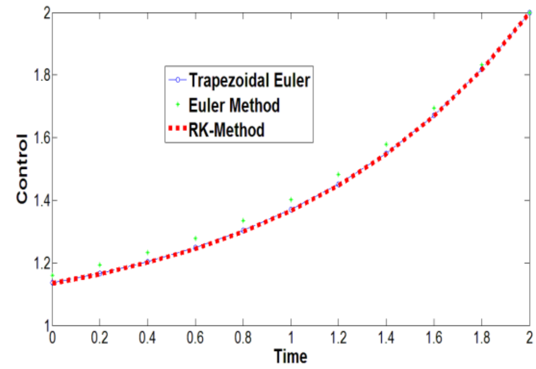

Fig. 8: Optimal Control values at h=0.2

Optimal control problem with control constraints

Let us consider the test problem

$$\max\ x(4) - \int_0^4 u^2(t)dt$$

Subject to

$$x'(t) = x(t) + u(t), \qquad x(0) = 0$$

$$u(t) \leq 5$$

Construct Hamiltonian

$$H = -u^2(t) + \lambda(x + u)$$

The payoff term $\phi = x(4)$ is not included in the Hamiltonian, so only change in necessary conditions is in the transversality condition. Specifically, since $\phi(x) = x$ and $\phi' = 1$, we have

Then, the adjoint calculation yields

$$\lambda'(t) = -\frac{\partial H}{\partial x} = -\lambda, \quad \lambda(4) = \varphi' = 1$$

As we know that $\frac{\partial H}{\partial u} < 0$implies u^* is at the lower bound, but we have no lower bound in this problem. To find a representation of u^*, we consider only two cases.

$$\frac{\partial H}{\partial u} = \lambda - 2u$$

$$\begin{cases} u \leq 5 & if\ \dfrac{\partial H}{\partial u} = 0 \\ u = 5 & if\ \dfrac{\partial H}{\partial u} > 0 \end{cases}$$

Our resulting optimality system is

$$x'(t) = x(t) + u(t), \qquad x(0) = 0$$

$$\lambda'(t) = -\lambda, \quad \lambda(4) = \varphi' = 1$$

We take TOL: 0.001

Table2: OPTIMAL CONTROL VALUES EVALUATED BY RK-METHOD, EULER AND TRAPEZOIDAL METHOD AT H=0.004

Selected values of Time	FBSM with Euler's	FBSM with Trapezoidal	FBSM with R-K
0.0000	4.9982	4.9976	4.9976
0.0200	4.9982	4.9976	4.9976
0.1400	4.9982	4.9976	4.9976
0.2880	4.9982	4.9976	4.9976
0.3400	4.9982	4.9976	4.9976
0.4200	4.9982	4.9976	4.9976
0.6400	4.9982	4.9976	4.9976
1.9960	3.7209	3.7075	3.7075
2.2120	2.9973	2.9873	2.9873
2.6120	2.0080	2.0024	2.0024
3.3040	1.0040	1.0024	1.0024
4.0000	0.4999	0.4998	0.4998

Similarly we use Euler, trapezoidal and Rung-e-Kutta methods with FBSM for solving above system. And check out the results at different step sizes.

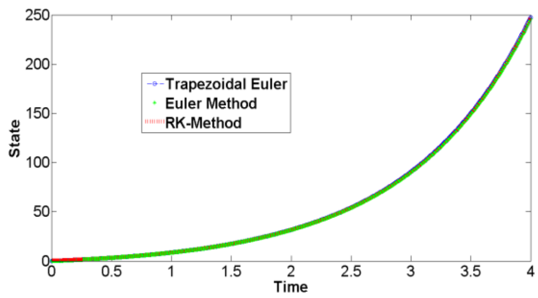

Fig. 9: Optimal state values at h=0.004

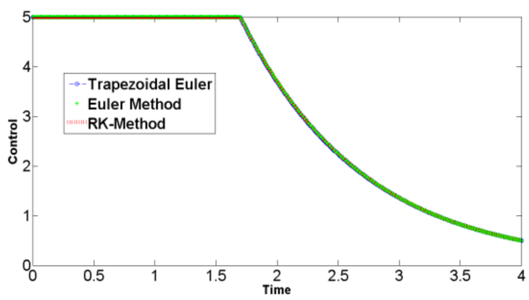

Figure10: Optimal control values at h=0.004

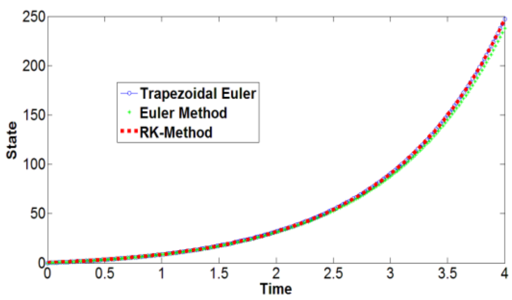

Fig. 11: Optimal state values at h=0.02

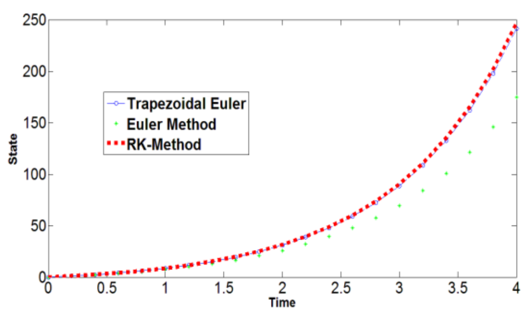

Fig. 15: Optimal state values at h=0.2

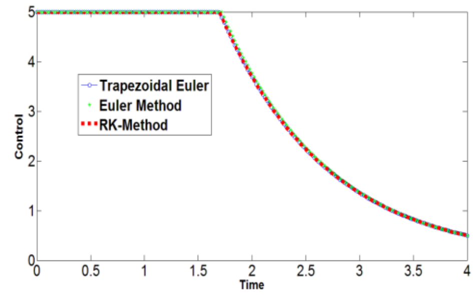

Fig. 12: Optimal control values at h=0.02

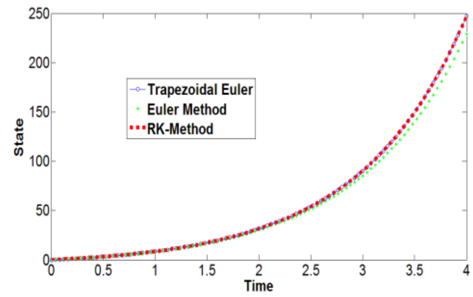

Fig. 13: Optimal state values at h=0.04

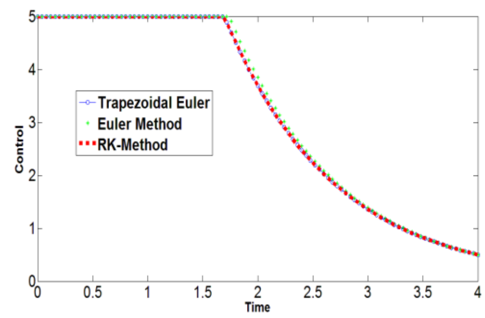

Fig. 14: Optimal control values at h=0.04

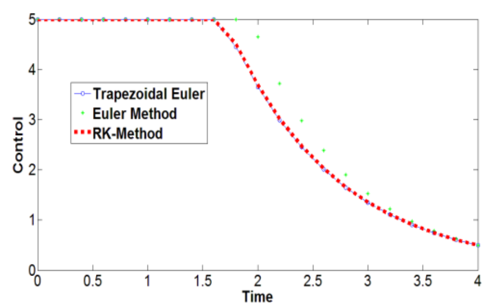

Fig. 16: Optimal control values at h=0.2

4. DISCUSSIONS

We use FBSM to solve bounded and unbounded control problem with payoff term. We modify the algorithm of FBSM to solve such optimal control problems. When we put constraints on controls we can attain desirables result in specific time interval. Also with payoff term we can maximize or minimize objective functional at terminal or final time as minimize the number of infected individuals at final time in an epidemic time. As we use Rung-e-Kutta, Euler and Trapezoidal schemes, it is also observed that the FBSM with Rung-e-Kutta and Trapezoidal schemes produce almost same result as compared to values of FBSM with Euler's scheme. And this difference increases when step size increases. In future

1) Convergence of Forward-Backward Sweep Method can be proved.
2) Using Optimal Control theory, one can adjust control in a system like Partial differential equations.
3) The basic problem can be generalized in the terminal value of the state x (T) may be fixed.
4) Optimal control theory can applied to various fields like Economics, Engineering and Biological-systems.

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